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1

Mathematical Morphology for Complete Lattices

J. SERRA

1.1 BASIC PROPERTIES OF LATTICES

Let us look briefly at the basic properties and definitions of lattices. The reader will find a more complete study in Birkhoff (1983) and Dubreil and M.L. Dubreil-Jacotin (1964). A sup (resp. inf) semilattice is an ordered set $\mathscr{P}$ in which any two elements X_1 and X_2 have a smallest majorant (resp. greatest minorant) $X_1 \vee X_2$ (resp. $X_1 \wedge X_2$) called the upper bound or sup (resp. lower bound or inf). The terms sup and inf are abbreviations for the Latin words supremum and infimum. The semilattice is complete if each family of elements $X_i \in \mathscr{P}$, finite or not, has an upper bound (resp. lower bound), thus implying the existence of a greatest element (resp. least element) called the universal element U (resp. null element $\emptyset$).

A set that is both a sup and an inf semilattice is called a lattice. If the semilattices are complete then so is the lattice. Any complete semilattice with universal and null elements is a complete lattice. In a lattice any logical consequence of a choice of ordering remains true when we commute the symbols $\vee$ and $\wedge$ or $<$ and $>$. This is called *the principle of duality w.r.t. order*.

The relation "$X < Y$" means X is smaller than Y and the symbol "$Y > X$" means Y is larger than X.

A Moore family is a part $\mathscr{B}$ of a complete lattice $\mathscr{P}$, where

(1) the universal element belongs to $\mathscr{B}$: $U \in \mathscr{B}$;

(2) for all non-empty parts $\mathscr{A} \subset \mathscr{B}$ we have

$$\wedge \{B : B \in \mathscr{A}\} \in \mathscr{B}$$

If $\mathscr{B}_0$ is a family in $\mathscr{P}$ then the class closed under $\wedge$, generated by $\mathscr{B}_0$, united with the universal element U, constitutes a Moore family. Following our previous reasoning, a Moore family is a complete lattice. Given a family $\mathscr{B}$ of this type, there exists one and only one extensive, increasing, idempotent mapping ϕ whose invariant sets coincide with $\mathscr{B}$. This mapping is called an

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algebraic closing or *Moore closing*, and ϕ and $\mathcal{B}$ are equivalent notions. In the theory of morphological filtering we make great use of this operation (see Chapter 5 in particular) and its dual, the algebraic opening; in the latter, the family of invariant sets is closed under sup and contains the null element.

Substituting an algebraic point of view, we can present a lattice as a set $\mathcal{P}$ equipped with two laws of composition $\vee$ and $\wedge$. Each law is
commutative: $X \wedge Y = Y \wedge X$,
associative: $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$,
and they are linked by the law of absorption:

$$X \wedge (X \vee Y) = X; \quad X \vee (X \wedge Y) = X.$$

These three axioms are the equivalent of a lattice associated with an ordering. If we restrict ourselves to the inf operation (for example) and we replace the last axiom by idempotence, i.e. $X \wedge X = X$, then the axiomatic system is equivalent to an inf semilattice.

Finally, we define a *chain* as any completely ordered subset in a lattice. If ψ is an increasing mapping of lattice $\mathcal{P}$ into itself then the image by ψ of any chain in $\mathcal{P}$ is still a chain.

This summary will suffice for the following chapter; we shall find at the beginning of Chapter 2 several complementary notions concerning distributivity and complementation. For the rest of this chapter $\mathcal{P}$ will always denote a complete lattice (see also James and James, 1976).

Examples of non-Boolean lattices

One may consider this algebraic structure to be rather far from practical applications. In fact, this structure models the *most common* procedures in applied morphology. We can see this in three examples: real-valued functions ("grey-tone morphology"), which are constantly used in morphology; partitions, less frequent, but used in segmentation problems; and topologically open sets.

Throughout this work we shall return to these three examples (Sections 1.7, 2.1 and 3.8 and Chapters 5 and 6). The confrontation between the first two of these examples is instructive. Although the first lattice is rather intuitive, the second is much less so (in particular in sup). The first is distributive, the second is not even modular (cf. Section 2.1). Monotonic continuity is better expressed in the first than in the second. Finally, the first is situated at the level of pixels, the second is on the larger scale of classes of pixels.

(a) Lattices of u.s.c. functions Let us consider the space of upper-semi-continuous (u.s.c.) numerical functions, bounded or not, with values in $\mathbb{R}^n$. We shall associate with each function $f(x)$ of this class its *umbra* $U(f)$, i.e. the set of points $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$f(x) = \sup \{t : (x, t) \in U(f)\}.$$

The ordering relation $U(f) \subset U(g)$ is equivalent to

$$\forall x : f(x) \leq g(x).$$

Conversely, if $\mathcal{U}$ denotes the class of closed umbrae, i.e.

$$\mathcal{U} = \{U : U \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}), \forall (x, t) \in U : t' \leq t \Rightarrow (x, t') \in U\},$$

then to each element $U \in \mathcal{U}$ there corresponds one and only one u.s.c. function $f(x)$ for which U is the umbra (see Chapter 9). Since the class $\mathcal{U}$ is closed for intersection and for *finite* union, it constitutes a lattice for inclusion (Fig. 1.1). If we interpret this in terms of u.s.c. functions, then for all families (finite or not) of index i we have

$$f < g \Leftrightarrow U(f) \subset U(g) \Leftrightarrow \forall x \in \mathbb{R}^n, \ f(x) \leq g(x),$$
$$\bigwedge_i f_i = \{f : U(f) = \bigcap_i U(f_i)\},$$
$$\bigvee_i f_i = \{f : U(f) = \bigcup_i \overline{U(f_i)}\}.$$

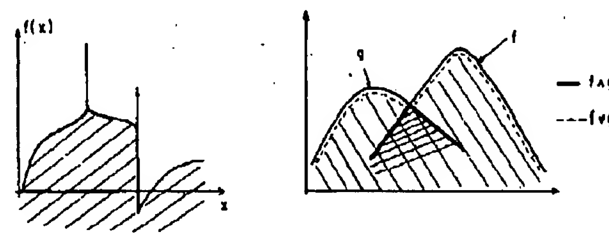

Fig. 1.1 (a) Umbra of an u.s.c. function. (b) Sup and inf of two functions.

(b) The partition lattice Let E be an arbitrary set. We define a partition of E as the division of the set into classes such that each point $x \in E$ belongs to one and only one of these classes. More formally, we can represent a partition T as a mapping of E into $\mathcal{P}(E)$ such that

(i) $\forall x \in E, \quad x \in T(x)$,
(ii) $\forall (x, y) \in E, \quad T(x) = T(y) \quad \text{or} \quad T(x) \cap T(y) = \emptyset$,

where the image $T(x)$ is the partition class that contains the point x. $\mathcal{T}$ will denote the family of partitions defined on E. We know (Simon, 1985) that we can associate the following ordering relation with $\mathcal{T}$:

$$T, T' \in \mathcal{T}: T < T' \Leftrightarrow T(x) \subset T'(x) \quad \forall x \in E.$$

A partition T is finer than T' when, for all x, the class $T(x)$ is included in $T'(x)$. In particular, the finest class of all has all the points of E as elements,

and the coarsest has only one element, namely the set E itself. Furthermore, this ordering allows us to construct a complete lattice. Let T_i, $i \in I$, be a family (finite or not) of partitions. We can easily see that the mapping $T : E \to \mathscr{P}(E)$, as defined by

$$T(x) = \bigcap_i T_i(x) \quad \forall x \in E,$$

generates a partition. But, by construction $T < T_i$; and for all $x \in E$, $T(x)$ is the largest element of $\mathscr{P}(E)$ that is contained in each $T_i(x)$. T is therefore the inf of T_i for ordering $<$, i.e.

$$T = \wedge_i T_i,$$

and the family $\mathscr{T}$, a complete inf semilattice with a universal element, is a complete lattice.

In this structure, the sup has a more complex expression than that of the inf. The expression $T = \vee T_i$ means that the partition T is the smallest partition that is larger than each T_i, or, for all x and for any i, the class $T(x)$ is the smallest set that is a union of classes $T_i(y)$, $y \in E$. Formally,

$$T = \bigvee_{i \in I} T_i \Leftrightarrow T(x) = \bigcap \{B, \forall i \in I : B = \bigcup_{y \in B} T_i(y), x \in B, B \in \mathscr{P}(E)\}.$$

In contrast with $\wedge T_i$, $T_i \in \mathscr{T}$, which has no particular reason to be a trivial partition, $\vee T_i$ is frequently equal to the universal element E (Fig. 1.2b).

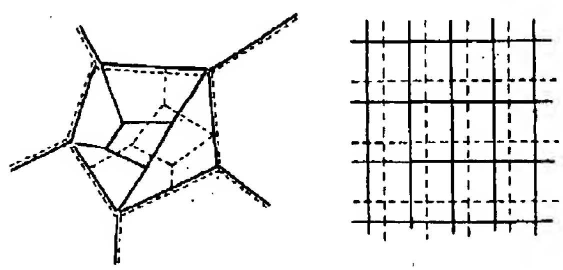

Fig. 1.2 (a) Two partitions of the plane into polygons. In the places where the two classes coincide, they constitute the corresponding class in the sup. If this is not the case then we must continue until a "smallest common multiple" is found. (b) Two partitions (one is just the translation of the other) whose sup is $\mathbb{R}^2$ itself.

(c) Lattices of open sets Let E be a topological space, and $\mathscr{G}$ its family of open sets. We can associate the following sup and inf with set inclusion:

$$X = \bigcup \{X_i, i \in I\}, \quad X_i \in \mathscr{G},$$
$$Y = \overline{\bigcap \{X_i, i \in I\}}^{\circ}, \quad X_i \in \mathscr{G}.$$

The set $\mathscr{G}$ is therefore a complete lattice (by duality for complementation we see that $\mathscr{F}$, the set of closed sets of E, is also a complete lattice).

Note that in the three examples (a), (b) and (c) there are no elements of the lattices that could be defined as points.

1.2 DILATION AND EROSION IN A COMPLETE LATTICE

We are going to study the class $\mathscr{S}(\mathscr{P})$, or more briefly $\mathscr{S}$, of mappings of $\mathscr{P}$ into itself that commute with the sup operation, such that

$$(1.1) \qquad \Gamma(\vee X_i) = \vee\Gamma(X_i), \quad i \in I, \quad X_i \in \mathscr{P},$$

and in particular $\Gamma(\emptyset) = \emptyset$. Following the notation already introduced in Euclidean mathematical morphology, we shall call this type of operation *dilation* (see Serra, 1982a). The structure of relation (1.1) shows us that dilation is an increasing operation. Moreover, any mapping of class $\mathscr{S}$ is accompanied by an equivalent transformation, called erosion, and characterized by the following theorem

Theorem 1.1 *An increasing mapping Γ belongs to the class $\mathscr{S}$ iff we can associate with it another mapping $\dot{\Gamma} : \mathscr{P} \to \mathscr{P}$ such that*

$$(1.2) \qquad \Gamma(X) < Y \Leftrightarrow X < \dot{\Gamma}(Y), \quad X, Y \in \mathscr{P}.$$

The mapping $\dot{\Gamma}$ is thus unique and increasing, and is given by

$$(1.3) \qquad \dot{\Gamma}(X) = \vee\{B : B \in \mathscr{P}, \Gamma(B) < X\}.$$

Proof We shall first show that (1.2) ⇒ (1.3) by considering the mapping

$$\Gamma^*(X) = \vee\{B : B \in \mathscr{P}, \Gamma(B) < X\}.$$

$\Gamma(B) < X \Rightarrow B < \dot{\Gamma}(X)$; thus $\Gamma^*(X) < \dot{\Gamma}(X)$. Conversely $\dot{\Gamma}(X) < \dot{\Gamma}(X)$, thus by (1.2): $\Gamma \circ \dot{\Gamma}(X) < X$ and $\dot{\Gamma}(X) < \Gamma^*(X)$. This gives us $\Gamma^* = \dot{\Gamma}$ and proves uniqueness, since if $\dot{\Gamma}$ exists it must be of the form (1.3).

Now let us look at (1.2) ⇒ (1.1). Let $\{X_i\}$ be a family of elements of $\mathscr{P}$. Applying (1.2) by taking X_i for X and $\vee\Gamma(X_i)$ for Y, we have

$$\Gamma(X_i) < \bigvee_i \Gamma(X_i) \Leftrightarrow \forall i, X_i < \dot{\Gamma}(\bigvee_i \Gamma(X_i)) \Leftrightarrow \vee X_i < \dot{\Gamma}(\vee\Gamma(X_i))$$
$$\Leftrightarrow \Gamma(\vee X_i) < \vee\Gamma(X_i).$$

But we also find the inverse inclusion, and, since Γ is increasing, (1.1) follows.

Conversely (1.1) ⇒ (1.2). To any mapping $\Gamma \in \mathscr{S}$, we can associate its erosion $\dot{\Gamma}$, as defined by (1.3). The form of (1.3) shows that $\Gamma(B) < X \Rightarrow B < \dot{\Gamma}(X)$.

For the inverse implication, since Γ commutes with sup, we have $B < \dot{\Gamma}(X) \Rightarrow \Gamma(B) < \Gamma\dot{\Gamma}(X) = \vee[\Gamma(B), B \in \mathscr{P}, \Gamma(B) < X] < X$. Finally, the growth of $\dot{\Gamma}$ can be seen directly in relation (1.3). ■

Remark Since the only operation occurring explicitly in the above discussion was sup, one might think that the theorem uses only the axioms of semilattices. In fact the definition of erosion, (1.3), assumes the existence of the null element in $\mathscr{P}$ (X may not contain a $\Gamma(B)$); then $\mathscr{P}$ is a complete semilattice with null and universal elements ($\emptyset$ and U) and is thus a complete lattice (Dubreil and Jacotin-Dubreil, 1964 p. 175).

Properties of dilation and erosion

(a) Erosion commutes with the operation inf. Indeed, since $\dot{\Gamma}$ is increasing, we have $\dot{\Gamma}(\wedge X_i) < \wedge \dot{\Gamma}(X_i)$, but also

$$\dot{\Gamma}(X_i) > \wedge \dot{\Gamma}(X_i) \Leftrightarrow \forall i, \quad X_i > \Gamma(\wedge \dot{\Gamma}(X_i)) \Leftrightarrow X_i > \Gamma(\wedge \dot{\Gamma}(X_i))$$
$$\Leftrightarrow \dot{\Gamma}(\wedge X_i) > \dot{\Gamma}(X_i).$$

(b) $\dot{\Gamma}(U) = U$, because $\Gamma(U) < U \Leftrightarrow U < \dot{\Gamma}(U)$.

(c) The class $\mathscr{S}$ constitutes a semigroup and contains a neutral element, the identity I, since for each pair $(\Gamma_1\Gamma_2) \in \mathscr{S}$

$$(\Gamma_2\Gamma_1)(\vee X_i) = \Gamma_2(\vee\Gamma_1(X_i)) = \vee\Gamma_2\Gamma_1(X_i).$$

Iterating, we see that $\Gamma_3(\Gamma_2\Gamma_1) = (\Gamma_3\Gamma_2)\Gamma_1$. We then deduce that erosions also form a semigroup since

$$(\Gamma_2\Gamma_1)(X) = \vee[B : B \in \mathscr{P}, (\Gamma_2\Gamma_1)(B) < X]$$
$$= \vee[B : B \in \mathscr{P}, \Gamma_1(B) < \dot{\Gamma}_2(X)] = \dot{\Gamma}_1\dot{\Gamma}_2(X).$$

(d) If Γ_1 and Γ_2 are ordered then so are $\dot{\Gamma}_1$ and $\dot{\Gamma}_2$, but in the opposite sense. Thus

$$[\forall X : \Gamma_i(X) < \Gamma_j(X)] \Leftrightarrow [\forall X : B < \dot{\Gamma}_j(X) \Rightarrow B < \dot{\Gamma}_i(X)]$$
$$\Leftrightarrow [\forall X : \dot{\Gamma}_j(X) < \dot{\Gamma}_i(X)].$$

(e) The class $\mathscr{S}$ has the structure of a complete lattice. If suffices to equip $\mathscr{S}$ with an ordering relation

$$\underset{\text{in } \mathscr{S}}{\Gamma_i < \Gamma_j} \Leftrightarrow \underset{\text{in } \mathscr{P}}{\Gamma_i(X) < \Gamma_j(X)} \quad \forall X \in \mathscr{P}.$$

We then find the equality $(\vee\Gamma_i)(X) = \vee\Gamma_i(X)$, whose second member belongs to $\mathscr{S}$, since

$$\bigvee_i \Gamma_i(\vee X_k) = \bigvee_i \bigvee_k \Gamma_i(X_k) = \bigvee_k \bigvee_i \Gamma_i(X_k).$$

$\mathscr{S}$ is therefore a sup-semilattice; now $\mathscr{S}$ has a least element $\Gamma_0(X) = \emptyset$, $\forall X \in \mathscr{P}$. Consequently, if Γ_i is a family in $\mathscr{S}$ then the set of lower bounds $\mathscr{M}$ is not empty since it contains Γ_0. As $\mathscr{S}$ is a complete sup-semilattice, $\mathscr{M}$ must have a greatest element m, which is the greatest lower bound of the Γ_i. $\mathscr{S}$ is therefore a complete lattice with inf

$$\inf[\Gamma_i : \Gamma_i \in \mathscr{S}] = \vee[M : M \in \mathscr{M}] = m.$$

In contrast with the sups in $\mathscr{P}$ and $\mathscr{S}$, which are directly linked, the infs are not linked at all.

(f) Dilation and erosion are increasing mappings. The derivation of the former follows directly from the form of (1.1), in which it is defined. As to the latter, we deduce it from (1.3), since

$$X_1 < X_2 \Rightarrow \Gamma(X_1) < \Gamma(X_2), \quad \dot{\Gamma}(X_1) < \dot{\Gamma}(X_2).$$

We are going to show that this property has a converse and that any increasing mapping ψ of $\mathscr{P}$ into itself such that $\psi(U) = U$ can be represented as the sup of erosions. Let $\psi : \mathscr{P} \to \mathscr{P}$ be an increasing mapping. Given

$$\Gamma_B(Y) = \begin{cases} \emptyset & \text{if } Y = \emptyset, \\ B & \text{if } Y < \psi(B), \\ U & \text{if } Y \text{ is not included in } \psi(B). \end{cases}$$

$\Gamma_B(Y)$ satisfies relation (1.1). It is therefore a dilation. To determine the corresponding erosion, we must first distinguish the case where $X > B$ from the contrary in the expression

$$(1.4) \qquad \dot{\Gamma}_B(X) = \vee[Y : Y \in \mathscr{P}, \Gamma_B(Y) < X].$$

As with $\Gamma_B(X)$, the erosion $\dot{\Gamma}_B(X)$ can have only three values. Obviously, if $X = U$ then $\dot{\Gamma}_B(U) = U$. If $X \neq U$ and $X > B$ then the inequality $\Gamma_B(Y) < X$ is satisfied if and only if $Y < \psi(B)$; since the sup of these Y is $\psi(B)$, we have $\dot{\Gamma}_B(X) = \psi(B)$. If X is not larger than B then *either* $Y < \psi(B)$, giving $\Gamma_B(Y) = B$ and Y is not smaller than $\dot{\Gamma}_B(X)$, where $\dot{\Gamma}_B(X)$ is defined by (1.4), *or* Y is not smaller than $\psi(B)$, and since $X \neq U$ (otherwise it would be larger than B), $\Gamma_B(Y) = U$ is not smaller than X. Finally, $\dot{\Gamma}(X)$ is reduced to the null element:

$$\dot{\Gamma}_B(X) = \begin{cases} U & \text{if } X = U, \\ \psi(B) & \text{if } X > B \text{ and } X \neq U, \\ \emptyset & \text{otherwise.} \end{cases}$$

Taking the sup of the $\dot{\Gamma}_B(X)$ when B spans $\mathscr{P}$, and considering the growth of ψ, we have

$$\vee[\dot{\Gamma}_B(X), B \in \mathscr{P}] = \vee[\psi(B) : B < X] = \psi(X).$$

Conversely, since erosion is increasing, the upper bound of an arbitrary family of erosions is also increasing. We can now state the following

Theorem 1.2 *The class of mappings $\Gamma : \mathcal{P} \to \mathcal{P}$ of a complete lattice into itself, and that commute with sup, constitutes a complete lattice of increasing mappings. Moreover, any mapping $\psi : \mathcal{P} \to \mathcal{P}$ such that $\psi(U) = U$ is increasing iff it can take the form of an upper bound of erosions. More precisely, the mapping ψ can be written as*

$$\psi = \vee[\dot{\Gamma}_B, B \in \mathcal{P}],$$

with $\dot{\Gamma}_B(X) = \psi(B)$ if $X > B$ and $\dot{\Gamma}_B(X) = \emptyset$ otherwise.

1.3 MORPHOLOGICAL OPENINGS AND CLOSINGS

In general, the mappings $X \to \Gamma(X)$ and $X \to \dot{\Gamma}(X)$ do not have inverses, and there is no way to determine X from its images $\Gamma(X)$ or $\dot{\Gamma}(X)$. But this indeterminacy is only partial. We shall always have either an upper or a lower bound, depending on the situation at hand.

For example, consider $\dot{\Gamma}(X)$. This is, as we saw in (1.3), the sup of Bs whose images under Γ are smaller than X. Consequently, the sup of all these images constitutes the minimal inverse image of $\dot{\Gamma}(X)$. In more algebraic terms: if we take in (1.2) $\dot{\Gamma}(Y)$ for the set X, the inequality $\dot{\Gamma}(Y) < \dot{\Gamma}(Y)$ is equivalent to

$$\Gamma\dot{\Gamma}(Y) < Y \quad \forall Y \in \mathcal{P}.$$

In the same way, taking $Y = \Gamma(X)$ in (1.2), we find

$$X < \dot{\Gamma}\Gamma(X) \quad \forall X \in \mathcal{P}.$$

Using the generic symbols for openings and closings, γ and ϕ, let us write

$$\gamma_\Gamma = \Gamma\dot{\Gamma} \quad \text{and} \quad \phi_\Gamma = \dot{\Gamma}\Gamma.$$

Actually, γ_Γ and ϕ_Γ are idempotent. For example, $\dot{\Gamma}\Gamma > I$ implies $\gamma_\Gamma\gamma_\Gamma = \Gamma\dot{\Gamma}\Gamma\dot{\Gamma} > \Gamma\dot{\Gamma}$, but since γ_Γ is anti-extensive, we have $\gamma_\Gamma\gamma_\Gamma < \gamma_\Gamma$, and thus the desired result. To avoid confusion concerning openings and closings and the more general algebraic concepts attached to these terms (Section 1.1), we shall call them *morphological* (and remove the index Γ where there is no ambiguity). Explicitly,

$$\text{(1.5)} \qquad \begin{cases} \gamma(Y) = \gamma_\Gamma(X) = \vee[\Gamma(B), B \in \mathcal{P}, \Gamma(B) < X], \\ \phi(X) = \phi_\Gamma(X) = \vee[B, B \in \mathcal{P}, \Gamma(B) < \Gamma(X)]. \end{cases}$$

We know that algebraic openings (resp. closings) are characterized by their invariant domains (cf. Sections 1.1 and 5.4). In the present case it is just the

form of the defining algorithm, $\gamma = \Gamma\dot{\Gamma}$, that shows that if B is open by γ, i.e. if $\gamma(B) = B$, then B is of the type $B = \Gamma(Z)$. Conversely, if Z is an arbitrary element of $\mathcal{P}$ then by extensivity of closing $\gamma\Gamma(Z) = \Gamma\dot{\Gamma}\Gamma(Z) > \Gamma(Z)$, and since γ is an opening, we have $\gamma\Gamma(Z) < \Gamma(Z)$. Thus the family $\mathcal{B}_\gamma$ of invariant sets of the morphological opening γ is the image $\Gamma(\mathcal{P})$ of $\mathcal{P}$ under Γ. Similarly, for the closing we find $\mathcal{B}_* = \dot{\Gamma}(\mathcal{P})$, since $Z \in \mathcal{P}$ implies $\dot{\Gamma}(\Gamma\dot{\Gamma})(Z) < \dot{\Gamma}(Z)$, because $\Gamma\dot{\Gamma}$ is an opening, and also $(\dot{\Gamma}\Gamma)\dot{\Gamma}(Z) > \dot{\Gamma}(Z)$, since $\dot{\Gamma}\Gamma$ is a closing, and finally $\phi\dot{\Gamma}(Z) = \dot{\Gamma}(Z)$. To summarize, we have the following.

Theorem 1.3 *The products $\gamma = \Gamma\dot{\Gamma}$ and $\phi = \dot{\Gamma}\Gamma$ define respectively morphological opening and closing on the lattice $\mathcal{P}$. The class of invariant sets of the former is the image of $\mathcal{P}$ under Γ, and that of the latter forms the image of $\mathcal{P}$ under $\dot{\Gamma}$.*

Algebraic openings and morphological openings

We can easily see that any upper bound of morphological openings γ_i, with corresponding domains of invariance $\mathcal{B}_i$, is yet another opening, but in this case not morphological. It has as its domain of invariance the class closed under the sup (in $\mathcal{P}$) generated by the union of the $\mathcal{B}_i$ (in the space $\mathcal{P}(\mathcal{P})$ of the subets of $\mathcal{P}$). Moreover, this property is valid for any opening, and is a classical result of the theory of morphological filtering (see Section 5.4).

Now consider the converse problem. Starting with an arbitrary opening $\gamma : \mathcal{P} \to \mathcal{P}$, we can consider it as a sup of morphological openings? We know from Matheron (1975) and Proposition 5.3 of Chapter 5 that any algebraic opening γ on $\mathcal{P}$, with invariant domain $\mathcal{B}$, is the smallest extension to $\mathcal{P}$ of the identity on $\mathcal{B}$, and is written

$$\text{(1.6)} \qquad \gamma(X) = \vee[B : B \in \mathcal{B}, B < X] \quad \forall X \in \mathcal{P}.$$

The class $\mathcal{B}$ is closed under sup. Conversely, the class closed under union generated by an arbitrary class, $\mathcal{B}_0 \in \mathcal{P}(\mathcal{P})$, defines, with the aid of (1.6), a mapping that is an opening. Now, associate with each $B \in \mathcal{B}$ the dilation

$$\Gamma_B(A) = B \quad \text{if } A \text{ not} < B; \qquad \Gamma_B(A) = \emptyset \quad \text{if } A < B.$$

Its corresponding morphological opening is

$$\gamma_B(X) = \vee[\Gamma_B(A) : \Gamma_B(A) < X] = \begin{cases} B & \text{if } B < X, \\ \emptyset & \text{if } B \text{ not} < X, \end{cases}$$

and (1.6) is consequently equivalent to

$$\gamma = \vee[\gamma_B, B \in \mathcal{B}].$$

In other words, we have the following.

Theorem 1.4 *If the mapping $\psi : \mathcal{P} \to \mathcal{P}$ is an opening then it has a representation of the form*

$$(1.6') \qquad \psi = \vee[\gamma_B, B \in \mathcal{B}],$$

where γ_B is the morphological opening associated with the dilation $\Gamma_B(A) = \emptyset$ if $A < B$ and $\Gamma_B(A) = B$ otherwise, and where $\mathcal{B}$ is the domain of invariance of ψ. Conversely, if $\mathcal{B}$ denotes the class closed under union generated by $\mathcal{B}_0$, a class of arbitrary elements of $\mathcal{P}$, then the mapping defined by (1.6') is an opening.

Size distribution

Matheron (1975) proposed an axiomatic definition of what one intuitively understands by size distribution (see Section 5.5). His definition brings into play families of (algebraic) openings, γ_λ, depending on a positive parameter λ such that

$$(1.7) \qquad \lambda \geq \mu \Rightarrow \gamma_\lambda = \gamma_\mu\gamma_\lambda = \gamma_\lambda\gamma_\mu.$$

We propose to characterize the family of dilations Γ_λ, whose openings can be used to generate size distributions. With this aim, consider the class of Γ_λ such that

$$(1.8) \qquad \lambda \geq \mu > 0 \Rightarrow \Gamma_\lambda = \gamma_\mu\Gamma_\lambda.$$

To each element $Y \in \mathcal{P}$, associate its erosion $X = \dot{\Gamma}_\lambda(Y)$ and apply the algorithm (1.8). Thus

$$\Gamma_\lambda\dot{\Gamma}_\lambda(Y) = \gamma_\mu\Gamma_\lambda\dot{\Gamma}(Y), \qquad \text{i.e. } \gamma_\lambda = \gamma_\mu\gamma_\lambda.$$

Moreover, idempotence implies $\gamma_\lambda < \gamma_\mu\gamma_\lambda\gamma_\mu\gamma_\lambda < \gamma_\lambda\gamma_\mu$, and we have, by growth, $\gamma_\lambda > \gamma_\lambda\gamma_\mu$. So (1.8) implies (1.7).

Conversely, apply the first equality from (1.7) to the element $\Gamma_\lambda(X), X \in \mathcal{P}$. We find $\Gamma_\lambda\dot{\Gamma}_\lambda\Gamma_\lambda = \gamma_\mu\Gamma_\lambda\dot{\Gamma}_\lambda\Gamma_\lambda$. But on the one hand, we have

$$\Gamma_\lambda < \Gamma_\lambda(\dot{\Gamma}_\lambda\Gamma_\lambda) = \gamma_\mu(\Gamma_\lambda\dot{\Gamma}_\lambda)\Gamma_\lambda < \gamma_\mu\Gamma_\lambda,$$

and, on the other, we also have

$$\Gamma_\lambda > (\Gamma_\lambda\dot{\Gamma}_\lambda)\Gamma_\lambda = \gamma_\mu\Gamma_\lambda(\dot{\Gamma}_\lambda\Gamma_\lambda) > \gamma_\mu\Gamma_\lambda,$$

and (1.7) implies (1.8), which *characterizes* the classes Γ_λ that can generate size distributions. We notice that no ordering relation of the Γ_λ was required. In summary, we have the following.

Theorem 1.5 *The morphological openings associated with any family $[\Gamma_\lambda]$ of dilations depending on a positive parameter λ generate a size distribution over the lattice $\mathcal{P}$ iff for any λ, Γ_λ is morphologically open by the γ_μ where $\mu \leq \lambda$, that is to say, $\lambda \geq \mu$ implies $\Gamma_\lambda = \gamma_\mu\Gamma_\lambda$.*

Remarks

(i) We can easily see that the axiom (1.7) is equivalent to ordering the (morphological or algebraic) openings by λ:

$$(1.7) \Leftrightarrow [\lambda \geq \mu > 0 \Rightarrow \gamma_\lambda < \gamma_\mu].$$

(ii) In the same manner, we could have introduced size distributions from their invariant sets. We shall see in Section 5.5 that the axiom (1.7) is also equivalent to

$$\lambda \geq \mu > 0 \Rightarrow \mathcal{B}(\gamma_\lambda) \subset \mathcal{B}(\gamma_\mu).$$

(iii) Taking the dual point of view, we similarly define antisize distributions as being families of algebraic closings $[\phi_\lambda]$, depending on the positive parameter λ, such that one of the following three equivalent axioms is satisfied:

$$\lambda \geq \mu > 0 \Rightarrow \phi_\lambda\phi_\mu = \phi_\mu\phi_\lambda = \phi_\lambda,$$
$$\lambda \geq \mu > 0 \Rightarrow \phi_\lambda > \phi_\mu,$$
$$\lambda \geq \mu > 0 \Rightarrow \mathcal{B}(\phi_\lambda) \subset \mathcal{B}(\phi_\mu).$$

(iv) As an example of sizing families, $[\Gamma_\lambda]$, we can take semigroups of type $\Gamma_\lambda = \Gamma_\mu\Gamma_{\lambda-\mu}$, $\lambda \geq \mu > 0$, which will often be used in the sequel. We can elaborate discrete versions of them, which are nonetheless size distributions, from iterations of dilations $\Gamma^{(k)}$ of order k, since by associativity $\Gamma^{(n)} = \Gamma^{(k)}\Gamma^{(n-k)}$ (k, $n - k$ positive integers) (see Section 4.3).

1.4 DUAL ASPECT, THE POINT OF VIEW OF EROSIONS

If until now we have given priority to the concept of dilation, it was for pedagogical reasons. We established an initial set of results without too much consideration of the fact that each one has its analogue based on erosion. It goes without saying that the symmetry of the roles of sup and inf in lattice $\mathcal{P}$ hold for all the logical constructions built upon it, and this leads us to looking at the five preceding theorems form the view point of duality. Consider the first of these, (1.1). To do this, consider the class of operations called erosions, $\mathcal{S}'$, which commute with inf, i.e. such that

$$(1.9) \qquad \Gamma^*(\wedge_i X_i) = \wedge_i \Gamma^*(X_i), \qquad i \in I, \quad X_i \in \mathcal{P},$$

and such that $\Gamma^*(U) = U$. Saying that Γ^* belongs to $\mathcal{S}'$ is equivalent to saying that there exists a dilation $\Gamma_1 \in \mathcal{S}$ such that

$$(1.2) \qquad \Gamma_1(X) < Y \Leftrightarrow X < \Gamma^*(Y) \quad \forall \text{ pairs } (X, Y) \in \mathcal{P}.$$

This dilation is unique, and is given by

$$(1.10)\qquad \Gamma_1(X) = \wedge\{B : B \in \mathcal{P}, \Gamma^*(B) > X\}.$$

Conversely, if we start with an arbitrary dilation Γ, the relation (1.2) produces a corresponding erosion $\dot\Gamma$ that is unique and commutes with inf. The preceding discussion tells us that we can use the algorithm (1.10) to determine the associated dilation Γ_1. But since

$$\Gamma_1(X) = \wedge\{B : B \in \mathcal{P}, X < \dot\Gamma(B)\} = \wedge\{B, B \in \mathcal{P}, \Gamma(X) < B\} = \Gamma(X),$$

it follows that Γ^* and Γ' coincide. There is an *isomorphism* between $\mathcal{S}$ and $\mathcal{S}'$, and the latter is also a complete lattice. From this we obtain the following.

Theorem 1.6 *Let $\mathcal{P}$ be a complete lattice, $\mathcal{S}$ the class of mappings $\Gamma : \mathcal{P} \to \mathcal{P}$ that commute with sup, and $\mathcal{S}'$ the class of mappings $\dot\Gamma$ that commute with inf. $\mathcal{S}$ and $\mathcal{S}'$ are two complete isomorphic lattices, which correspond to one another through the duality relation*

$$\Gamma(X) < Y \Leftrightarrow X < \dot\Gamma(Y) \quad \forall X \in \mathcal{P}, \quad Y \in \mathcal{P}.$$

To each dilation $\Gamma \in \mathcal{S}$ there corresponds an erosion $\Gamma' \in \mathcal{S}'$:

$$\dot\Gamma(X) = \vee\{B : B \in \mathcal{P}, \Gamma(B) < X\},$$

and to each erosion $\dot\Gamma \in \mathcal{S}'$ there corresponds a dilation $\Gamma \in \mathcal{S}$:

$$\Gamma(X) = \wedge\{B : B \in \mathcal{P}, \dot\Gamma(B) > X\}.$$

Remarks

(i) This theorem suggests a tautological expression of Γ and $\dot\Gamma$. For example, each Γ can be written as

$$\Gamma(X) = \wedge\{B : B \in \mathcal{P}, \vee\{C, C \in \mathcal{P}, \Gamma(C) > B\} > X\}.$$

(ii) Theorem (1.2) is easily transposed, and any increasing mapping ψ such that $\psi(\emptyset) = \emptyset$ can be interpreted as an inf of dilations. In the same manner, any algebraic closing can be interpreted as an inf of morphological closings (Theorem 1.4), and any family of erosions $\{\Gamma'_\lambda\}$ defines an antisize distribution ϕ_λ if $\lambda \geq \mu$ implies $\dot\Gamma = \phi_\mu\Gamma'_\lambda$ (Theorem 1.5). Theorem 1.3 remains unchanged.

1.5 MONOTONE CONTINUITY IN $\mathcal{P}$, $\mathcal{S}$ AND $\mathcal{S}'$

If we wish to remain in the general algebraic framework of the lattice $\mathcal{P}$, and nevertheless wish to describe operations such as size distributions, then a

certain amount of continuity is very useful. All that is available to us is *monotone continuity*. We can express this at two levels of generality. Given an increasing mapping, $\psi : \mathcal{P} \to \mathcal{P}$, then either, for each increasing sequence $\{X_n\}$, n an integer, we have

$$\psi(\vee X_n) = \vee\psi(X_n)$$

(which is written $X_n \uparrow X \Rightarrow \psi(X_n) \uparrow \psi(X)$), and we are then working with *sequential* monotone continuity; or, for each filtering family $\{X_i\}$, $i \in I$, where I is an ordered set and $i < j \Rightarrow X_i < X_j$, we have

$$\psi(\vee X_i, i \in I) = \vee\psi(X_i),$$

and we are then dealing with *general* monotone continuity. The latter is much stronger than the former (note once again that $X_i \uparrow X \Rightarrow \psi(X_i) \uparrow \psi(X)$).

Obviously, by duality these notions extend to cover decreasing sequences as well as decreasing mappings. Between the sequential case where n spans the set of positive integers $\mathbb{Z}^+$, and the general case of the ordered space I, we find the case where the index λ of X_λ describes a subset of $\mathbb{R}^+$. However, this case is equivalent to sequential monotone continuity. Indeed, if the family $\{\lambda\}$ is bounded above by λ_0, and X_λ increases w.r.t. λ, then

$$X_{\lambda_0} = \vee\{X_\lambda : \lambda < \lambda_0\} \text{ is equivalent to } X_{\lambda_0} = \vee_n X_{\lambda_0 - 1/n}.$$

If the increasing mapping ψ satisfies sequential monotone continuity, i.e.

$$\psi(X_{\lambda_0}) = \psi(\vee_n X_{\lambda_0 - 1/n}) = \vee_n \psi(X_{\lambda_0 - 1/n}),$$

then it follows that $\psi(X_{\lambda_0}) = \vee\{\psi(X_\lambda) : \lambda < \lambda_0\}$ and sequential monotone continuity is sufficient. As it is obviously necessary (evident if we begin with $X'_\lambda = \vee\{X_n : n < \lambda\}$), it is *equivalent* to the monotone continuity in λ on $\mathbb{R}^+$.

In the following pages we shall rarely use general monotone continuity, and we shall always specify it explicitly. Otherwise, we shall speak of $\downarrow$ continuity or $\uparrow$ continuity, and similarly we shall say that a mapping is *continuous* when it is both $\uparrow$ and $\downarrow$ continuous.

Relation (1.1), which defines the class of dilations, implies their $\uparrow$ continuity in the general sense, and, by duality, the $\downarrow$ continuity of erosions. It follows from this that if Γ is $\downarrow$ continuous (and thus continuous) then $\gamma = \Gamma\dot\Gamma$ and $\phi = \dot\Gamma\Gamma$ are also $\downarrow$ continuous, since $\dot\Gamma$ is always $\downarrow$ continuous, by duality γ and ϕ are $\uparrow$ continuous when $\dot\Gamma$ is.

However, $\downarrow$ continuity for Γ does not imply $\uparrow$ continuity for $\dot\Gamma$. In general, neither openings nor closings are continuous. In summary, we have the following.

Theorem 1.7 *Dilation is a $\uparrow$ continuous mapping of $\mathcal{P}$ into itself and erosion is a $\downarrow$ continuous mapping. Beyond that, if dilation Γ is continuous*

then opening $\Gamma\dot{\Gamma}$ and closing $\dot{\Gamma}\Gamma$ are $\downarrow$ continuous. In the same manner, if $\dot{\Gamma}$ is continuous then $\Gamma\dot{\Gamma}$ and $\dot{\Gamma}\Gamma$ are $\uparrow$ continuous. All of these continuities must be understood in the sense of general monotone continuity.

Remark The use here of the term "continuous" is consistent with its topological sense. In fact, any chain in a lattice $\mathscr{P}$ (i.e. any completely ordered subset) constitutes a topological space for the ordering topology. The increasing mappings on $\mathscr{P}$, which transform a chain into a chain, therefore connect two topological spaces. We thus introduce a topology, but only if we restrict ourselves to monotone sequences. We shall see that for a great many problems this constraint is largely acceptable. In exchange, we shall be able to define concepts that would otherwise have been inaccessible. One example of this is the alternating sequential filter in Chapter 10 (this remark also applies to left- and right-continuities).

Left- and right-continuity

In the preceding discussion we assumed X to be variable and Γ fixed. However, certain concepts, such as size distribution, bring us to families of dilations $\{\Gamma_\lambda\}$, which depend on a scalar λ.

We shall say that the mapping $\lambda \to \Gamma$ from $\mathbb{R}^+$ into $\mathscr{S}$ is increasing if for all $X \in \mathscr{P}$ we have

$$\lambda \geq \lambda' \Rightarrow \Gamma_\lambda(X) > \Gamma_{\lambda'}(X),$$

and that it is decreasing if for all X

$$\lambda \geq \lambda' \Rightarrow \Gamma_\lambda(X) < \Gamma_{\lambda'}(X).$$

If the mapping $\lambda \to \Gamma_\lambda$ is increasing, we say $\Gamma_\lambda \downarrow \Gamma_{\lambda_0}$ if and only if $\lambda \downarrow \lambda_0$ implies $\Gamma(X) \downarrow \Gamma_{\lambda_0}(X)$ for all $X \in \mathscr{P}$ (resp. $\Gamma_\lambda \uparrow \Gamma\lambda_0$ when $\lambda \uparrow \lambda_0$). By duality, this property transfers to decreasing mappings $\lambda \to \Gamma_\lambda$.

To avoid confusion between the convergence of $\Gamma_\lambda(X)$ with X variable or with λ variable, we apply the term left-continuity (or right-continuity) to mappings, $\lambda \to \Gamma_\lambda$, $\lambda \to \dot{\Gamma}_\lambda$, etc. Specifically, Γ_λ is left-continuous when $\lambda \uparrow \lambda_0$ implies $\Gamma_\lambda \uparrow \Gamma_{\lambda_0}$ (or $\Gamma_\lambda \downarrow \Gamma_{\lambda_0}$). We obtain right-continuity by starting at $\lambda \downarrow \lambda_0$. Finally, for mappings such as $(\lambda, X) \to \Gamma_\lambda(X)$ or $(\lambda, X) \to \gamma_\lambda(X)$, which combine variations in X and λ, we shall use the general terminology $\uparrow$ and $\downarrow$ continuity.

If the mapping $\lambda \to \Gamma_\lambda$ is increasing and left-continuous then its dual $\lambda \to \dot{\Gamma}_\lambda$ is decreasing and left-continuous. We showed it to be decreasing in Section 1.2, (b) on p. 18. To establish left-continuity, start with the following equivalences for all pairs $X, Y \in \mathscr{P}$:

$$\begin{aligned}\Gamma_{\lambda_0} = \vee\{\Gamma_\lambda(X) < Y : \lambda < \lambda_0\} &\Leftrightarrow \forall\lambda < \lambda_0,\ \Gamma_\lambda(X) < Y\\ &\Leftrightarrow \forall\lambda < \lambda_0,\ X < \dot{\Gamma}_\lambda(Y)\\ &\Leftrightarrow X < \wedge\{\dot{\Gamma}_\lambda(Y);\ \lambda < \lambda_0\}.\end{aligned}$$

Thus

$$\dot{\Gamma}_{\lambda_0}(X) = \wedge\{\dot{\Gamma}(X) : \lambda < \lambda_0\} \quad \forall X \in \mathscr{P}.$$

Note that in general, a right-limit of dilations (as defined by an inf) need not be a dilation.

Semicontinuity and size distributions

Proposition 1.8 *If the dilation $\Gamma_\lambda : \mathscr{P} \to \mathscr{P}$ is continuous for all λ and the mapping $\lambda \to \Gamma_\lambda$ is increasing and left-continuous then the size distribution $(\lambda, X) \to \gamma_\lambda(X)$ that maps $\mathbb{R}^+ \times \mathscr{P}$ into $\mathscr{P}$ is $\downarrow$ continuous.*

Proof

Given two sequences, $\lambda \uparrow \lambda_0$ and $X_\rho \downarrow X_{\rho_0}$, and setting $X_\rho = X_{\rho_0}$, we have not only

$$\begin{aligned}\gamma_{\lambda_0}(X_{\rho_0}) &= \Gamma(\wedge\, \dot{\Gamma}_\lambda(X_{\rho_0});\ \lambda < \lambda_0)\\ &= \wedge\,\{\Gamma_{\lambda_0}\dot{\Gamma}(X_{\rho_0});\ \lambda < \lambda_0\} > \wedge\,\{\gamma_\lambda(X_{\rho_0});\ \lambda < \lambda_0\}\end{aligned}$$

but also $\gamma\lambda_0 < \gamma_\lambda$ for all $\lambda \leq \lambda_0$. Therefore $\gamma\lambda_0 < \wedge\lambda < \lambda_0\gamma_\lambda$, which gives us the equality. Since Γ_λ is continuous relative to X, we can write

$$\gamma_{\lambda_0}(X_{\rho_0}) = \wedge\{\gamma_\lambda(X_{\rho_0});\ \lambda < \lambda_0\} = \wedge\{\gamma_\lambda(X_\rho);\ \rho > \rho_0;\ \lambda < \lambda_0\}. \quad ∎$$

Remarks

(i) We could have obtained a similar result by assuming $\lambda \to \Gamma_\lambda$ to be right-continuous. In this case it would no longer be necessary to take the hypothesis of $\downarrow$ continuity for $X \to \Gamma_\lambda(X)$. The two conditions $\lambda \downarrow \lambda_0$ (eventually $\lambda_0 = 0$) and $X_\rho \uparrow X_{\rho_0}$ would then imply $\gamma\lambda(X_\rho) \uparrow \gamma_{\lambda_0}(X_{\rho_0})$. These two versions foreshadow the semicontinuous behaviour that we meet when dealing with spaces of closed and open sets $\mathscr{F}$ and $\mathscr{G}$ of a locally compact, separable, Hausdorff space (denoted below by LCS; cf. for examples Matheron (1975, p. 25) or (Serra, 1982a, Chapter III)).

(ii) As to invariant sets, left-continuity of Γ_λ is equivalent to the implication $\lambda \uparrow \lambda_0 \Rightarrow \mathscr{B}(\gamma_\lambda) \downarrow \mathscr{B}(\gamma_{\lambda_0})$.

(iii) We find the dual results for antisize distributions by applying the same hypothesis while exchanging "left" for "right" and reversing the arrows.

Extreme elements

When describing an element of $\mathcal{P}$ by studying its evolution under the action of a family of decreasing erosions or openings, it is interesting to look at the last step in this evolution; that is to say, when the set disappears. Mathematical morphology has introduced two related concepts to handle this: the ultimate erosion, for erosions, and the critical element, for openings.

(a) Ultimate erosions Suppose we have a family $\{\dot{\Gamma}_\lambda\}$ that is decreasing w.r.t a positive λ and completed by $\dot{\Gamma}_0 = I$. Let λ vary over the interval $[0, \lambda_{max}]$ and consider $\mathcal{P}'(\lambda_{max})$, the class of elements $X \in \mathcal{P}$ whose erosion $\dot{\Gamma}_{\lambda_{max}}(X)$ is empty. As $\dot{\Gamma}_0 = I$ and $\dot{\Gamma}_\lambda$ decreases as λ increases, there exists a value λ_0 for each $X \in \mathcal{P}'(\lambda_{max})$ such that

(1.11) $$\lambda < \lambda_0 \Rightarrow \dot{\Gamma}_\lambda(X) \neq \emptyset, \qquad \lambda > \lambda_0 \Rightarrow \dot{\Gamma}_\lambda(X) = \emptyset$$

or

$$\lambda_0(X) = \sup\{\lambda : \dot{\Gamma}_\lambda(X) \neq \emptyset\}, \qquad X \in \mathcal{P}'(\lambda_{max}).$$

We then call $\dot{\Gamma}_{\lambda_0}(X)$ the *ultimate erosion* of X w.r.t. the family $\{\dot{\Gamma}_\lambda\}$. When the mapping $\lambda \to \Gamma$ is left-continuous, this element is determined by the erosions of lower index, since

$$\dot{\Gamma}_{\lambda_0}(X) = \wedge\{\dot{\Gamma}_\lambda(X); \lambda < \lambda_0\}.$$

(b) Critical elements of a size distribution As above, we can associate with each size distribution its critical elements (Matheron 1975, p. 194). The set X is critical for $\lambda = \lambda_0$ when

$$\lambda < \lambda_0 \Rightarrow \gamma_\lambda(X) \neq \emptyset; \qquad \lambda > \lambda_0 \Rightarrow \gamma_\lambda(X) = \emptyset$$

and $\lambda_0 = \sup\{\lambda : \gamma_\lambda(X) \neq \emptyset\}$. When $\lambda \uparrow \lambda_0$ implies $\gamma_\lambda \downarrow \gamma_{\lambda_0}$, we have

$$\gamma_{\lambda_0}(X) = \wedge\{\gamma_\lambda(X); \lambda < \lambda_0\} = M,$$

and, by idempotence for opening, $\gamma_{\lambda_0}(M) = M$. Since we know that $\mathcal{B}_{\lambda_0} \subset \mathcal{B}_\lambda$ for $\lambda \leq \lambda_0$, M is invariant for all openings smaller than λ_0:

$$\lambda \leq \lambda_0 \Rightarrow \gamma_\lambda(M) = M, \qquad \lambda > \lambda_0 \Rightarrow \gamma_\lambda(M) = \emptyset.$$

Conversely, to each $M \in \mathcal{P}$ such that $\gamma_{\lambda_{max}}(M) = \emptyset$, there corresponds a $\lambda_0 \in [0, \lambda_{max}]$ such that M is critical for λ_0.

Thus, subject to families of transformations $\{\dot{\Gamma}_\lambda\}$ and $\{\gamma_\lambda\}$ that progressively reduce any $X \in \mathcal{P}$ to the null element, these families produce two new mappings of $\mathcal{P}$ into $\mathbb{R}^+$: the ultimate erosion and the critical opening. We interpret them as descriptors of the maximum sizes for the elements of $\mathcal{P}$.

1.6 THE SEMIGROUPS $\Gamma_{\lambda+\mu} = \Gamma_\lambda\Gamma_\mu$

Amongst the classes of dilations depending on a positive parameter $\lambda \in \mathbb{R}^+$, those used most often are semigroups of the type

(1.12) $$\Gamma_\lambda\Gamma_\mu = \Gamma_{\lambda+\mu} \quad (0 < (\lambda, \mu) < \infty) \quad \text{and} \quad \Gamma_0 = I$$

They have the following properties.

(a) If $\lambda > \mu$ then Γ_λ is invariant under γ_μ. In this case the relation (1.12) is equivalent to $\Gamma_\lambda = \Gamma_\mu\Gamma_{\lambda-\mu}$, and, following from Theorem (1.3), $\Gamma_\lambda(X)$ is invariant under γ_μ for all $X \in \mathcal{P}$.

(b) Saying that Γ_λ is extensive is equivalent to saying that the mapping $\lambda \to \Gamma_\lambda$, from $\mathbb{R}^+$ into $\mathcal{S}$, is increasing. $X < \Gamma_\lambda(X)$ implies $\Gamma_\mu(X) < \Gamma_\mu\Gamma_\lambda(X) = \Gamma_{\lambda+\mu}(X)$; conversely, the implication $\lambda \geq \mu \Rightarrow \Gamma_\lambda > \Gamma_\mu$ has a special consequence for $\mu = 0$, the inclusion $\Gamma_\lambda > I$. Note that property (b) is satisfied when the definition (1.12) is replaced by the following weaker relation:

(1.13) $$\Gamma_\lambda\Gamma_\mu < \Gamma_{\lambda+\mu} \quad (\lambda, \mu > 0), \qquad \Gamma_0 = I.$$

(c) If the dilation $\Gamma_\lambda : \mathcal{P} \to \mathcal{P}$ is continuous for all $\lambda > 0$ and if $\lambda \downarrow 0$ implies $\Gamma_\lambda \downarrow \Gamma_0 = I$ then the mapping $\lambda \to \Gamma_\lambda : \mathbb{R}^+ \to \mathcal{S}$ is right-continuous. In fact, we find for all positive λ

$$\Gamma_\lambda = \Gamma_\lambda(\wedge_{\mu>0}\Gamma_\mu) = \wedge_{\mu>0}\Gamma_\lambda\Gamma_\mu = \wedge_{\mu>0}\Gamma_{\lambda+\mu}.$$

(d) If the family of dilations Γ_λ is increasing w.r.t. λ and if there exists a value $\lambda_0 > 0$ such that Γ_{λ_0} is left-continuous w.r.t. λ_0 then Γ_λ is left-continuous for all $\lambda > \lambda_0$.

(e) If a family of dilations Γ_ϵ is defined for the domain $[0, 2\epsilon_0]$ and satisfies the semigroup relation for all pairs $(\epsilon, \epsilon') \in [0, \epsilon_0]$ then we can construct an extension of $\mathbb{R}^+$ that retains the semigroup structure (1.12). For each real $\lambda \in \mathbb{R}^+$, let n be its integer quotient by ϵ_0. Set

(1.14) $$\Gamma^*_\lambda = (\Gamma_{\lambda/(n+1)})^{n+1}$$

The dilation Γ^*_λ is the iteration of order $n+1$ of a dilation of the family $\{\Gamma_\lambda\}$, since, by construction, $\lambda/(n+1) \in [0, \epsilon_0]$. We then use the commutativity of semigroups to see that $\Gamma^*_\lambda\Gamma^*_\mu = \Gamma^*_{\lambda+\mu}$.

(f) If the dilation Γ_ϵ defined in (e) is right-continuous w.r.t. ϵ for $\epsilon \in [0, \epsilon_0]$ then Γ^* is right-continuous for all $\lambda \in \mathbb{R}^+$. Also, if Γ_ϵ is left-continuous for $\epsilon \in [0, \epsilon_0]$ then Γ^*_λ is left-continuous for $\lambda \in [0, \infty]$ (which follows immediately from (1.14)).

(g) Let λ be a parameter taking values in the closed half-line $[0, \infty]$, and let Γ_λ be a family of extensive dilations of $\mathcal{P} \to \mathcal{P}$ such that

$$\Gamma_\lambda\Gamma_\mu < \Gamma_{\lambda+\mu}, \qquad \text{with } \Gamma_0 = I.$$

[redacted]

The quantities $d(X, Y) = \inf\{\lambda : X < \Gamma_\lambda(Y), Y < \Gamma_\lambda(X)\}$ (if we can find a λ that permits these inclusions) and $d(X, Y) = \infty$ (if we cannot) define a pseudometric on $\mathscr{P}$. The first two axioms for the pseudometric,

$$d(X, Y) = d(Y, X) \quad \text{and} \quad d(X, X) = 0$$

are obvious. For the third, the triangle inequality, let

$$\lambda_1 = d(Y, Z), \qquad \lambda_2 = d(X, Z), \qquad \lambda_3 = d(X, Y).$$

If λ_1 or $\lambda_3 = \infty$ then $\lambda_2 \le \lambda_1 + \lambda_3$. If not, then from the inclusions

$$X < \wedge\{\Gamma_\lambda(Y) : \lambda > \lambda_3\} = \Gamma_{\lambda_3+\epsilon}(X), \qquad Y < \Gamma_{\lambda_3+\epsilon}(X)$$

and

$$Y < \Gamma_{\lambda_1+\epsilon}(Z), \qquad Z < \Gamma_{\lambda_1+\epsilon}(Y)$$

we find

$$X < \Gamma_{\lambda_3+\epsilon}\Gamma_{\lambda_1+\epsilon}(Z) < \Gamma_{\lambda_1+\lambda_3+2\epsilon}(X),$$

and also

$$Z < \Gamma_{\lambda_1+\epsilon}\Gamma_{\lambda_3+\epsilon}(X) < \Gamma_{\lambda_1+\lambda_3+2\epsilon}(X),$$

from which we get $\lambda_2 \le \lambda_1 + \lambda_3 + 2\epsilon$ for all $\epsilon > 0$; therefore

$$\lambda_2 \le \lambda_1 + \lambda_3.$$

Generally speaking, when the product $\Gamma_\lambda\Gamma_\mu$ is commutative, the first pseudometric leads to an infinity of others, which are ordered amongst themselves. It suffices to take

$$\Gamma_\lambda^{(2)} = \Gamma_{\lambda/2} \circ \Gamma_{\lambda/2} < \Gamma_\lambda.$$

We have $\Gamma_0^{(2)} = I$ and

$$\Gamma_\lambda^{(2)}\Gamma_\mu^{(2)} = \Gamma_{\lambda/2}\Gamma_{\lambda/2}\Gamma_{\mu/2}\Gamma_{\mu/2} < \Gamma_{(\lambda+\mu)/2}\Gamma_{(\lambda+\mu)/2} = \Gamma_{\lambda+\mu}^{(2)}.$$

Each iteration of this process gives us a pseudometric that is weaker than the preceding one.

(h) If, beyond the hypothesis of (g), we allow

$$d(X, Y) < \infty \quad \forall X, Y \in \mathscr{P}$$

and

$$\Gamma_\lambda \downarrow \Gamma_0 = I \quad \text{when } \lambda \downarrow 0$$

then the quantity $d(X, Y)$ defines a *metric* on $\mathscr{P}$. It is clear that $d(X, Y) = 0$ is equivalent to

$$X < \wedge\{\Gamma_\lambda(Y); \lambda > 0\} \quad \text{and} \quad Y < \wedge\{\Gamma_\lambda(X); \lambda > 0\};$$

that is, because of the hypothesis of right-continuity for Γ_λ at $\lambda = 0$,

$$X < Y \quad \text{and} \quad Y > X.$$

1.7 EXAMPLES

To illustrate the major theorems of this chapter, we shall again take the three examples of lattices that were introduced in Section 1.1: namely the u.s.c. functions, the partitions and the open (or closed) Euclidean sets.

(a) Lattice $\mathscr{U}$

Amongst the various erosions that we can construct on $\mathscr{U}$, the most frequently used are those called Euclidean (cf. Chapter 5) and defined as follows. Let K be a compact set of $\mathbb{R}^n \times \bar{\mathbb{R}}$, let $K_{x,t}$ be its translation by the point (x, t), and let $\check{K}$ be the symmetric set of K with respect to the origin, i.e. $\check{K} = \{(-x, -t) : (x, t) \in K\}$. Given

$$U[f \ominus \check{K}] = \{(x, t) : K_{x,t} \subset U(f)\}$$

we can easily see that

$$U(f \ominus \check{K}) = \bigcap_{(x,t)\in \check{K}} [U(f)]_{x,t}.$$

Consequently, $U(f \ominus \check{K})$ is an umbra (and we denote the associated u.s.c. function by $f \ominus \check{K}$), and furthermore the mapping $f \to f \ominus \check{K}$ commutes with inf. We therefore have an *erosion* (cf. Fig. 1.3). By applying Theorem 1.6, we see that its dual dilation is given by

$$U(f \ominus K) = \{(x, t) : \check{K}_{x,t} \cap U(f) \neq \emptyset\} = \bigcup_{(x,t)\in K} [U(f)]_{x,t}$$

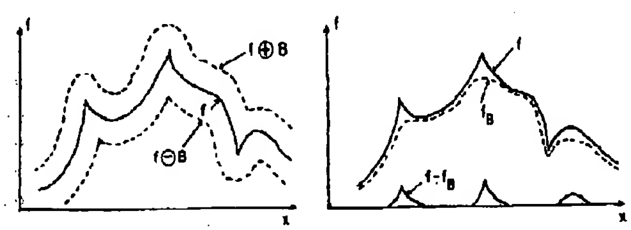

Fig. 1.3 (a) Erosion and dilation of function f by disc B. (b) Morphological opening f_B and the difference $f - f_B$ (the latter operation, called the "top-hat transform" (Meyer, 1978) or the "Rolling ball" (Sternberg, 1979), is a contrast algorithm frequently employed in morphology).

(the right-hand side of this equation is a union of closed sets—it is also topologically closed; Matheron, 1975, p. 19). From these expressions, we can deduce the morphological opening and closing. They are traditionally denoted by

$$f_K = (f \ominus \check{K}) \oplus K \quad \text{and} \quad f^{\check{K}} = (f \oplus K) \ominus \check{K}.$$

The set $U(f_K)$ is the zone of space in $\mathbb{R}^n \times \bar{\mathbb{R}}$ that is swept by the translation of K when they are included in $U(f)$. Here we have a geometrical interpretation of the first relation in (1.5) (there is a dual interpretation for the closing $f^{\check{K}}$).

The most useful families of operators are those obtained by starting with homothetic sets λK of a compact convex set K. In fact, the dilations Γ_λ defined by

$$\Gamma_\lambda(f) = f \oplus \lambda K \quad (K \text{ a compact and convex set})$$

satisfy (1.8), thus introducing size distributions, and also satisfy the semi-group relation (1.12). Amongst many relations concerning continuity, we have, for K a compact set,

$$\lambda \uparrow \mu \Rightarrow \begin{cases} f \oplus \lambda K \uparrow f \oplus \mu K, & f^{\lambda \check{K}} \uparrow f^{\mu \check{K}}, \\ f \ominus \lambda \check{K} \downarrow f \ominus \mu \check{K}, & f_{\lambda K} \downarrow f_{\mu K}, \end{cases}$$

and

$$f_i \downarrow f \Rightarrow \begin{cases} f_i \oplus K \downarrow f \oplus K, & (f_i)^{\check{K}} \downarrow f^{\check{K}}, \\ f_i \ominus \check{K} \downarrow f \ominus \check{K}, & (f_i)_K \downarrow f_K. \end{cases}$$

These relations can be interpreted as particular cases of another topology, which is more general than that of the present case ($\mathbb{R}^n$ being a topological space) (see Matheron, 1975, p. 16).

(b) Lattice $\mathcal{T}$

These rather formal examples will be complemented with more realistic ones on segmentation in Section 4.8.

Example 1 We define in the lattice $\mathcal{T}$ of partitions of an arbitrary space E the mapping $\dot{\Gamma} : \mathcal{T} \to \mathcal{T}$ as follows:

$$\dot{\Gamma}(T) = \begin{cases} T \wedge T_0 & \text{for all } T \in \mathcal{T},\ T \neq E, \\ E & \text{if } T = E, \end{cases}$$

where T_0 is a given partition (Fig. 1.4). The operation $\dot{\Gamma}$, which obviously commutes with inf, is an erosion. If $T < T_0$ then $\Gamma(T) = T$; conversely, if $\dot{\Gamma}(T) = T$ then $T = E$ or $T = T \wedge T_0$, and therefore $T < T_0$. The invariant sets under $\dot{\Gamma}$ form the family $\{T : T < T_0\} \cup E$.

According to Theorem 1.6, the dual dilation Γ is expressed by

$$\Gamma(T) = \wedge\{B : B \in \mathcal{T}, \dot{\Gamma}(B) > T\},$$

which shows that if $T < T_0$ then $\Gamma(T) = T = \dot{\Gamma}(T)$. For $T \nless T_0$, there exists no set $B \neq E$ such that $\dot{\Gamma}(B) > T$, because we also have $\dot{\Gamma}(B) < T_0$. We then have $\Gamma(T) = E$. In summary,

$$\Gamma(T) = \begin{cases} T & \text{if } T < T_0, \\ E & \text{if } T \nless T_0. \end{cases}$$

Finally, the morphological closing $\dot{\Gamma}\Gamma$ and opening $\Gamma\dot{\Gamma}$ coincide respectively with dilation and erosion.

Although the opening is too trivial to produce instructive size distributions, the erosion-closing $\dot{\Gamma}$, on the other hand, brings us to antisize distributions. It suffices to parametrize T_0 by $\lambda > 0$ so as to have

$$\lambda > \mu \Rightarrow T_0(\lambda) < T_0(\mu) \quad \text{and} \quad \lambda = 0 \Rightarrow T_0 = E,$$

and then

$$\dot{\Gamma}(T) = T \wedge T_0(\lambda)$$

is an antisize distribution.

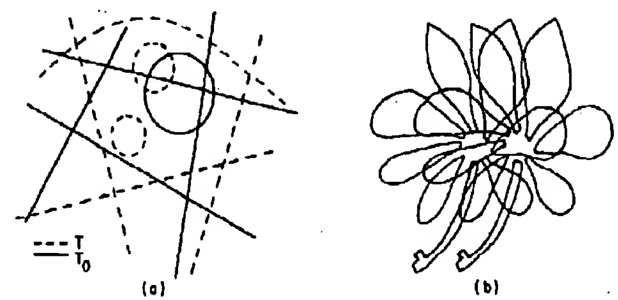

Fig. 1.4 Erosions of partitions: (a) example 1; (b) example 2.

Example 2 Now let us consider the lattice $\mathcal{T}$ of partitions of a vector space E. Given a partition $T \in \mathcal{T}$, let T_h denote the partition obtained by translating each class of T by a vector h. Now set

$$\dot{\Gamma}(T) = T \wedge T_h,$$

where h is a given vector. $\dot{\Gamma}$ is clearly an erosion. The corresponding dilation is obtained by noting that $B \wedge B_h > T$ implies $B > T$ and $B_h > T$, i.e. $B > T_{-h}$, and

$$B > T, \quad B > T_{-h} \Rightarrow B > T \vee T_{-h},$$

but

$$\Gamma(T \vee T_{-h}) = (T \vee T_{-h}) \wedge (T \vee T_h) > T;$$

therefore

$$\Gamma(T) = T \vee T_{-h}.$$

(From this we can deduce openings and closings, but they present no particular interest.)

Example 3: Hierarchies The concept of indexed hierarchy in a partition was introduced by Simon (1985, p. 125). It is a complete chain in the set of the lattice of partitions, i.e. it begins at $\emptyset$ and terminates at E. In terms of mappings, a hierarchy can be interpreted as a succession of transforms of $\emptyset$ according to a family $[\psi_\lambda]$ of increasing mappings such that

(a) for a $\lambda_0 \geq 0$ we have $\psi_{\lambda_0}(\emptyset) = \emptyset$;
(b) for a $\lambda_1 > \lambda_0$ we have $\psi_{\lambda_1}(\emptyset) = E$;
(c) $\lambda_1 \geq \lambda \geq \mu \geq \lambda_0 \Rightarrow \psi_\lambda(\emptyset) > \psi_\mu(\emptyset)$.

It is wise to suppose that the level at which we begin has no affect on the law that controls the hierarchy's progression, i.e. $\lambda > \mu \Rightarrow \psi_\lambda \psi_\mu(\emptyset) = \psi_\lambda(\emptyset)$. Axiom (c) is then replaced by the condition that $[\psi]$ be an antisize distribution. We shall see an example of hierarchy in Section 4.8.

(e) Lattice of the open (resp. closed) sets in $\mathbb{R}^n$

Take Euclidean space $\mathbb{R}^n$ of dimension n, and for all $X \subset \mathbb{R}^n$ and all vectors

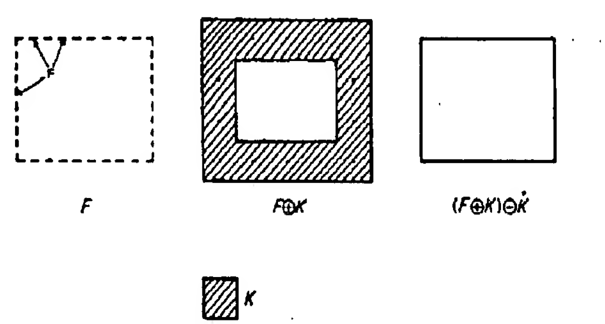

Fig. 1.5 Euclidean dilation and closing of a topologically closed set. (a) Series of segments that outline in dotted line the contour of a rectangle. (b) Dilation by the compact square K. (c) Morphological closing of F.

$h \in \mathbb{R}^n$, let X_h denote the set translation of X by h. We saw in Section 1.1 that the two sets $\mathcal{G}(\mathbb{R}^n)$, which are the topologically open sets, and $\mathcal{F}(\mathbb{R}^n)$, which are the topologically closed sets of $\mathbb{R}^n$, formed two complete lattices.

Consider the Minkowski addition $X \oplus K$ of set X by a structuring element K (see Section 3.2):

$$X \oplus K = \bigcup_{y \in K} X_y, \qquad X, Y \subset \mathbb{R}^n.$$

For a given K, $X \oplus K$ is clearly a dilation from $\mathcal{P}(\mathbb{R}^n)$ into itself. The corresponding erosion is written as

$$X \ominus \check{K} = \bigcap_{-y \in K} X_{-y}$$

It has been shown in Matheron (1975, Chapter 1) that if K is compact and X is open then the dilation $X \oplus K$, as well as the corresponding erosion, and the morphological opening and closing are all open sets. The same is true if K is compact and X is closed, for the four operations that map $\mathcal{F}(\mathbb{R}^n)$ into itself. The reader will find in Section 6.6 an example of the use of lattices $\mathcal{F}$ and $\mathcal{G}$.

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